OPTIMISATION AND MARGINAL ANALYSIS

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MATHEMATICAL RELATIONS

- Equations are mathematical description of how variables are related.
- In every equation, there is a dependent variable, independent variable and the functional notation

$$y = f(x)$$

- Dependent variable is the one that is explained in the model or whose value is determined by the model. It is called regressand or endogenous variable. In the eqation above 'y' is the dependent variable
- Independent variable is the one whose value is determined outside the model. It is also called regressor and exogenous variable. In the equation above, 'x' is the independent variable
- The function notation describes the nature of the underlying relationship between the dependent and independent variables. In the equation above, this is represented by 'f'

- The nature of the underlying relations between the dependent variable and independent variable can be linear and non-linear.

- Linear functions have constant slopes. Slopes measure the per unit change in the dependent variable as a result of a unit change in the independent variable. Linear functions can be uni-variate or multivariate .Examples of a linear function are

$$y = 2x$$
$$y = 20 + 0.5x$$
$$y = 15 + 2x + 0.6t + 3p$$

- Non-linear functions do not have constant slope. The slope differs from point to point. Examples are

$$y = 2x^2$$
$$y = 20 + 2x^2$$
$$y = 20 + 2x^2 + 0.5x^3$$

Rules of Differentiation

- Differentiation is the process of finding the derivative of a function

$$y = ax^n$$

$$\frac{dy}{dx} = f'(y) = nax^{n-1}$$

- **The constant function rule**:the derivative of a constant function is zero. Given $f(x) = k$; $f'(x) = 0$

- **The linear function rule:** the derivative of a variable raised to the first power is always equal to the coefficient of the variable.

$$y = ax + b$$

$$\frac{dy}{dx} = a$$

- Eg. $f(x) = 5 - \frac{1}{2}x$

- **The Power function rule:** the derivative of a power function is equal to product of the coefficient, say k and the exponent of the variable say 'x' raised to the exponent minus 1.

$$y = kx^n$$
$$\frac{dy}{dx} = f'(x) = n.k.x^{n-1}$$
$$f(x) = 5x^2$$
$$f(x) = 2 + x^4$$

- **The rules of sums and differences:** the derivative of a sum of two functions $f(x) = g(x) + h(x)$, where g(x) and h(x) are both differentiable functions, is equal to the derivatives of the individual functions. Similarly, the derivative of the differences of two functions is equal to the difference of the derivatives of the two functions

$$f'(x) = g'(x) + h'(x)$$
$$f'(x) = g'(x) - h'(x)$$
$$eg.f(x) = 12x^5 - 4x^4$$
$$f(x) = 9x^2 + 2x - 3$$
$$f(x) = 9x - 3x^4 + 4x^2$$

- **The Product rule**: the derivative of a product, where both are differentiable functions, is equal to the first function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first.

$$f(x) = g(x).h(x)$$
$$f'(x) = g(x).h'(x) + h(x).g'(x)$$
$$eg.f(x) = 3x^4(2x-5)$$
$$g(x) = 3x^4$$
$$h(x) = 2x-5$$

- **The Quotient rule:** the derivative of a quotient where both are differentiable functions, is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the denominator squared

$$f(x) = \frac{g(x)}{h(x)}$$
$$f'(x) = \frac{h(x).g'(x) - g(x).h'(x)}{[h(x)]^2}, h(x) \neq 0$$
$$f(x) = \frac{5x^3}{4x+3}; g(x) = 5x^3; h(x) = 4x+3$$

- **The Generalized power function rule:** the derivative of a function raised to a power, $f(x)=[g(x)]^n$, where g(x) is a differentiable function and n is any real number, is equal to the exponent n times the function g(x) raised to the power n-1, multiplied in turn by the derivative of the function itself g'(x)

$$f'(x)=n[g(x)]^{n-1}.g'(x)$$

$$eg.f(x)=(x^3+6)^5$$

- **The Chain Rule:** Given a composite function, also called function of a function, in which 'y' is a function of 'u' and 'u' is a function of 'x', i.e. y=f(u) and u=f(x), the derivative of y with respect to x is equal to the derivative of the first function with respect to u times the derivative of the second function with respect to x;

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$eg.y=(5x^2+3)^4$$

$$let, u=5x^2+3$$

$$y=u^4$$

- **Higher-order derivative:** the second-order derivative is the rate of change of the first derivative. Thus, higher-order derivatives are found by applying the rules of differentiation to lower-order derivatives

$$y = f(x)$$
$$f'(x)$$
$$f''(x)$$
$$f'''(x)$$
$$f(x) = 2x^4 + 5x^3 + 3x^2$$

- **Implicit differentiation:** implicit functions are those functions with both the dependent and independent variable occurring on side of the equal sign.

$$3x^4 - 7y^5 = 86$$
$$\frac{d(3x^4 - 7y^5)}{dx} = \frac{d(86)}{dx}$$
$$\frac{d(3x^4)}{dx} - \frac{d(7y^5)}{dx} = 0$$
$$12x^3 - 35y^4\frac{dy}{dx} = 0$$
$$12x^3 = 35y^4\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{12x^3}{35y^4}$$

Rules of Partial differentiation

- **Product rule**

$$z = g(x, y).h(x, y)$$

$$\frac{dz}{dx} = g(x, y).\frac{dh}{dx} + h(x, y)\frac{dg}{dx}$$

$$\frac{dz}{dy} = g(x, y).\frac{dh}{dy} + h(x, y).\frac{dg}{dy}$$

$$z = (3x + 5)(2x + 6y)$$

- **Quotient rule**

$$z = {}^{g(x, y)}\!\big/_{h(x, y)}$$

$$\frac{dz}{dx} = \frac{h(x, y)\frac{dg}{dx} - g(x, y)\frac{dh}{dx}}{[h(x, y)]^2}$$

$$\frac{dz}{dy} = \frac{h(x, y)\frac{dg}{dy} - g(x, y)\frac{dg}{dx}}{[h(x, y)]^2}$$

$$z = \frac{6x + 7y}{5x + 3y}$$

- **Generalised power function rule**

$$z = [g(x, y)]^n$$

$$\frac{dz}{dx} n[g(x, y)]^{n-1} . \frac{dg}{dx}$$

$$\frac{dz}{dy} n[g(x, y)]^{n-1} . \frac{dg}{dy}$$

$$z = (x^3 + 7y^2)^4$$

Illustration

- Given that demand for beef is $Q_D = 4850 - 5P_b + 1.5P_p + 0.1Y$, determine income elasticity of demand, cross price elasticity and own price elasticity if the price of beef is 200, income is 10,000 and the price of pork is 100
- The total product of a firm is given as $Q = 36KL - 2K^2 - 3L^2$, determine the marginal product of capital and labour

Relative extrema

- Relative extrema is the point at which a function is at a minimum or maximum

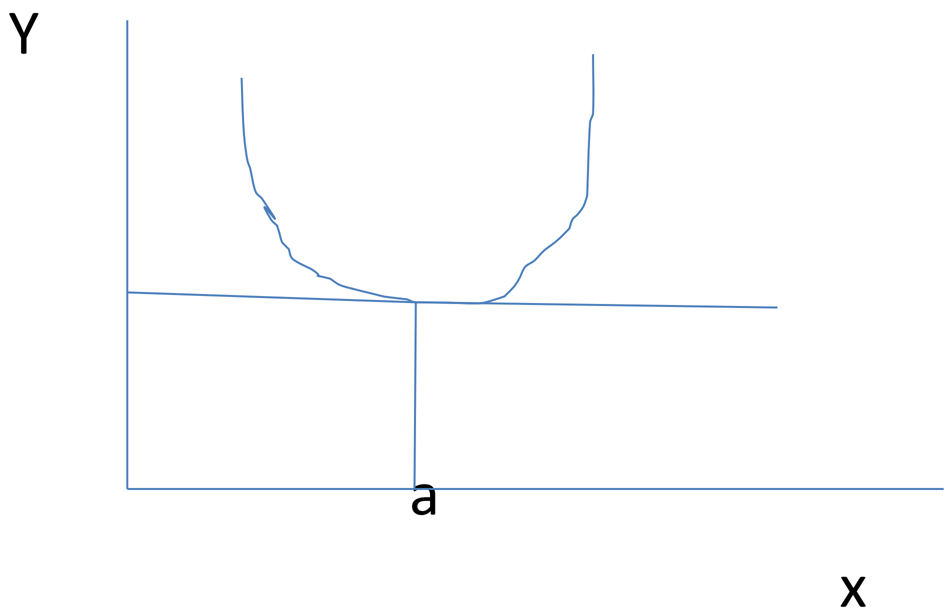
- To be at a relative maximum or minimum at a point ‘a’, the function must be at a relative plateau, i.e. neither increasing nor decreasing at ‘a’.

- If the function is neither increasing nor decreasing at ‘a’, the first derivative of the function at must equal zero or be undefined.

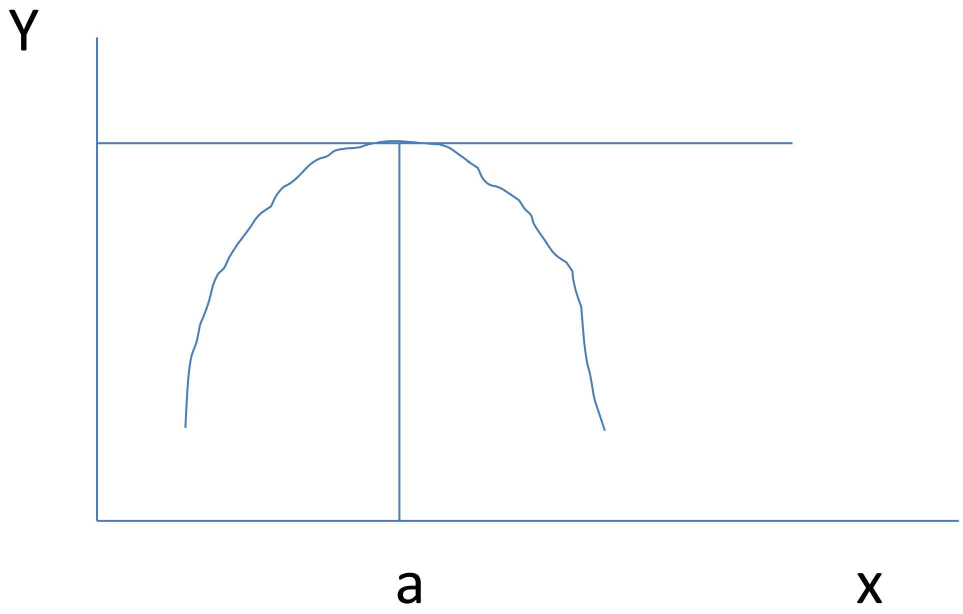
- A point in the domain of a function where the derivative equal zero or is undefined is called a critical point or value

- To distinguish mathematically between a relative maximum and minimum, the second-derivative is used. Assuming f'(a)=0,

- If f''(a)>0, the function is convex and the graph of the function lies completely above its tangent line at x=a, the function is at a minimum at x=a



- If f''(a)<0, the function is concave and the graph of the function lies completely below its tangent line at x=a, the function is at a relative maximum at x=a



- If f''(a)=0, the test is inconclusive

- Find the relative extrema for the following functions by (1) finding the critical values and (2) determining if at the critical value(s) the function is at a relative maximum or minimum

$$f(x) == 7x^2 + 126x - 23$$

$$f(x) = -(x-8)^2$$

Unconstrained optimisation

- The economist is frequently asked upon to help a firm maximize profit and physical levels of output and productivity, as well as minimize costs. This is done by determining the relative extrema

- Under unconstrained optimisation, it is assumed that the firm faces no constraint, in pursuit of the firm’s objective

Illustration

Suppose total and cost are given respectively as

$$R = 4000Q - 33Q^2$$

$$C = 2Q^3 - 3Q^2 + 400Q + 5000$$

- Determine the level of output at which profit is maximised. What is the maximum profit attained at this output level?

- Determine the level of output at which total revenue is maximised

- Given total output as $TP = 90K^2 - K^3$

1. Determine the level of k at which total output is maximised
2. Determine the level of k at which average product is maximised
3. Determine the level of k at which marginal product is maximised
4. Plot Total output, average product, and marginal product showing the relationship

- Given that the demand for a good is $Q = 1400 - P^2$, determine the elasticity of demand if p=20

- For each of the following total cost functions, find (1) the average cost function (2) the minimum average cost

$$TC = Q^3 - 5Q^2 + 60Q$$
$$TC = Q^3 - 21Q^2 + 500Q$$

- Assume a cost function of $C = Q^3 - 18Q^2 + 750Q$, Sketch a graph showing the relationship between total cost, average cost, and marginal cost.

- A firm producing two goods x and y has the profit function

$$\pi = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

Find the profit-maximising level of output for each of the two goods and test to be sure that profits are maximised.

- Assume a firm offers two different brands of a product, for which the demand functions are

$$Q_1 = 14 - 0.25P_1$$
$$Q_2 = 24 - 0.5P_2$$

and the joint cost of production is $TC = Q_1^2 + 5Q_1Q_2 + Q_2^2$, determine the profit-maximising level of output and the price that should charged to ensure this

Note: the sufficient condition for a multivariate function x=f(x,y) to be at an optimum is

$z_{xx}, z_{yy} > 0$ for minimum

$z_{xx}, z_{yy} < 0$ for maximum

$$z_{xx}z_{yy} > (z_{xy})^2$$

Constrained optimisation

- Differential calculus is also used to maximise or minimise a function subject to constraint
- Given a function f(x,y) subject to a constraint g(x,y)=k, a new function F can be formed

(1). Setting the constrained equal to zero, (2) multiplying it by λ (the langrange multiplier), and (3) adding the product to the original function

$$F(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)]$$

$F(x, y, \lambda)$ is the langrangian function

$f(x, y)$ is the objective function

$g(x, y)$ is the constraint function

- Critical values are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero, and solving for simultaneously

Illustration

Optimise $z = 4x^2 + 3xy + 6y^2$ subject to the constraint $x + y = 56$

- What combination of x and y should a firm produce to minimise costs subject to a production quota

$$c = 6x^2 + 10y^2 - xy + 30$$
$$x + y = 34$$

Significance of the langrange Multiplier

- It approximates the marginal impact on the objective function caused by a small change in the constrained. In utility maximisation subject to a constraint, the langrage multiplier will estimate the marginal utility of an extra dollar of income

Rules of indices

$$x^m . x^n = x^{m+n}$$

$$(x^m)^n = x^{m.n}$$

$$(xy)^m = x^m . y^m$$

$$x^{-m} = \frac{1}{x^m}$$

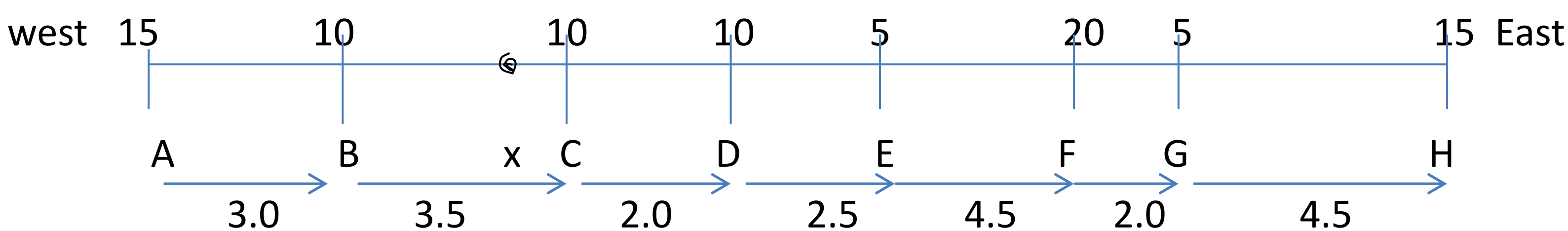
$$x^{\frac{1}{n}} = n\sqrt{x}$$

$$x^{\frac{m}{n}} = n\sqrt{x^m}$$

Marginal Analysis

- Marginal analysis is the process of considering small changes in a decision and determining whether a given change will improve the ultimate objective.

- Assume a real estate developer is contemplating on where to locate a shopping mall to minimise the total travel mile for customers. The figure below shows the number of customers , the various sites, and the corresponding travel times



- Suppose the estate developer chooses site x, what is the total travel mile
- What happens if the estate developer changes the location from x to c.

- What happens if the location is changed to D and E.
- Using total travel mile to identify the optimal site can be very involving and moreover, there is no guarantee that an optimal location can be found using this approach.
- The simplest way to deal with this problem is to apply the marginal analysis concept
- Using this approach, it can easily be deduced without much computation hassle that location E is the optimal location
- In real world business, firms are confronted with the decision of whether to employ one more unit of labour or capital, cut down price or increase price , increase advertising expenditure or not.
- By this the manager is interested in the impact of a marginal change in the decision variable on the objective variable. The concept of marginal analysis help managers arrive at the optimal decisions in these instances

A simple model of the firm

- A firm produces a single good or service for a single market with the objective of maximising profit
- Its task is to determine the quantity of the good to produce and sell and to set a sales price
- The firm can predict the revenue and cost with certainty

- Agya Appiah company ltd is occupied with determining the quantity of Agya Appiah bitters to produce and sell and the price

- To tackle this problem lets examine the profit function for this firm. Assume demand and cost of the product is given below

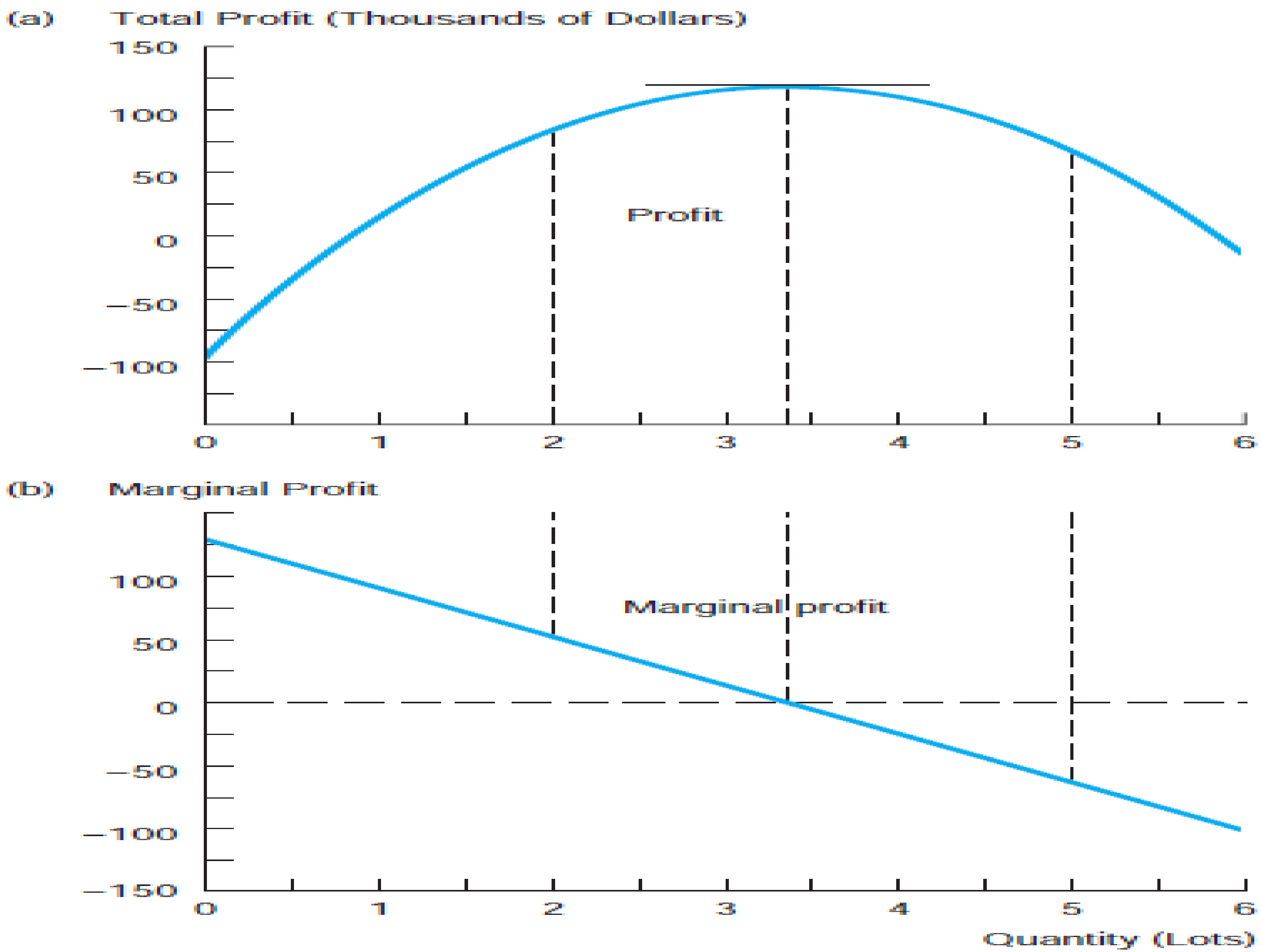
$$Q = 8.5 - 0.05P$$
$$C = 100 + 38Q$$

- PROFIT = REVENUE – COST

- REVENUE= PRICE * QUANTITY OF SALES

- First, one can use enumeration to arrive at the optimal level of profit, output and price (see illustration in class). This approach can be very tedious especially when you have very long data on sales.

- Using the marginal analysis, however, makes things relatively easy. Using this approach, we will look at the change in profit that result from making a small change in a decision variable (sales)
- Thus, this approach emphasise on the marginal profit, which is the change in profit resulting from a small increase in sales.


(a) Total Profit (Thousands of Dollars)
150
100
50
0
−50
−100
Profit
0
1
2
3
4
5
6
(b) Marginal Profit
100
50
0
−50
−100
−150
Marginal profit
0
1
2
3
4
5
6
Quantity (Lots)

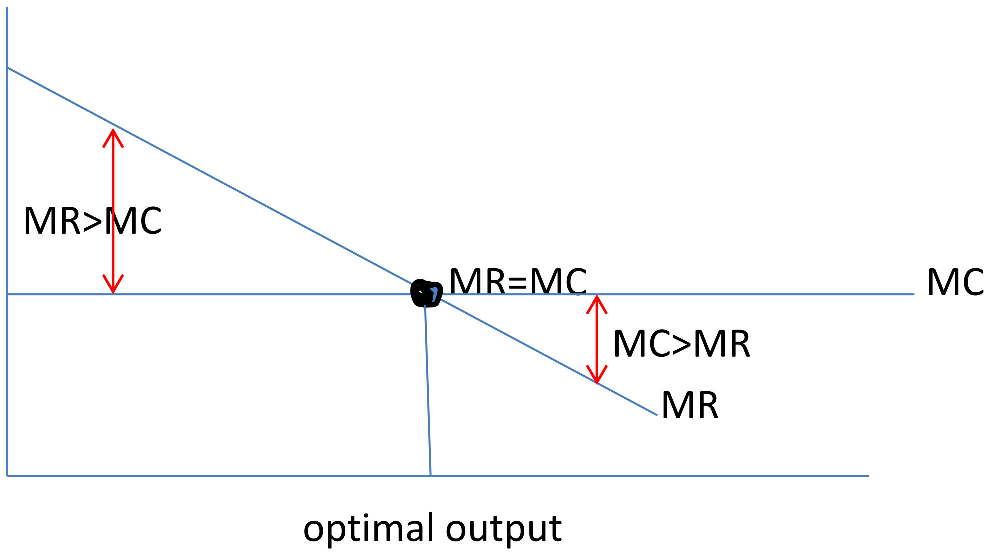
- Marginal analysis apart from providing an efficient tool for calculating the firm's optimal decision, helps identify the factors that determine profit

- Looking at the profit function, the key elements of profits are marginal cost and marginal revenue

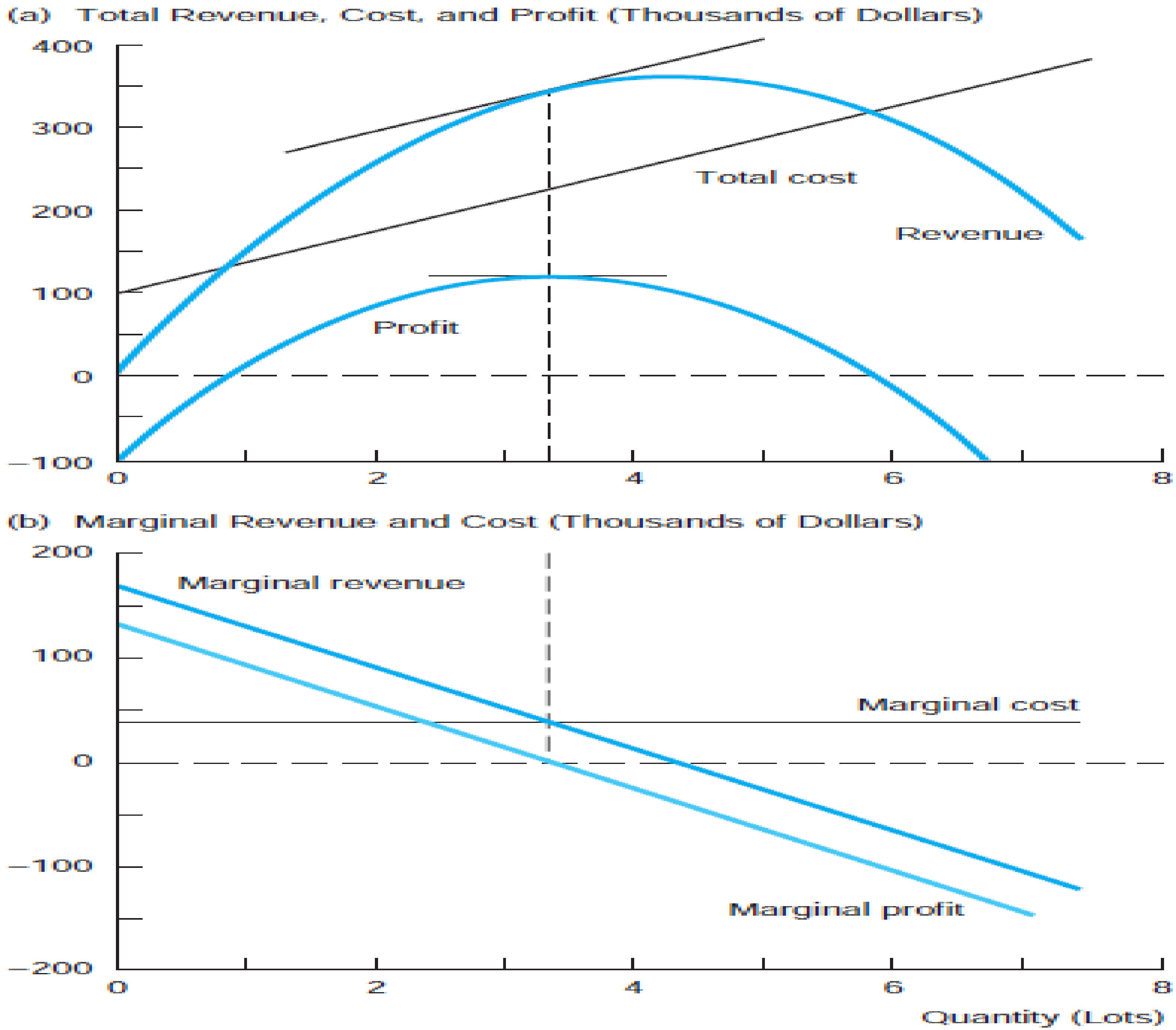
- The MR=MC is the shortest path to finding the firm's optimal output.

- Instead of finding the marginal profit and setting to zero, we simply take the MR and MC and equate the two.

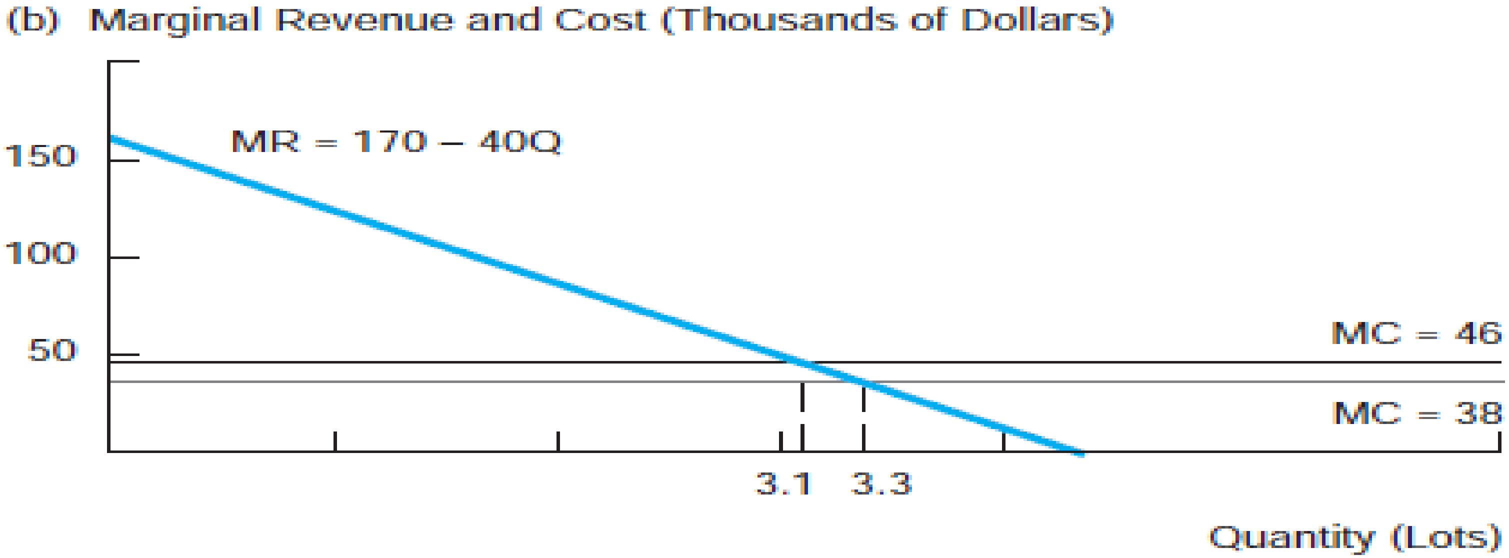
- Graphical illustration

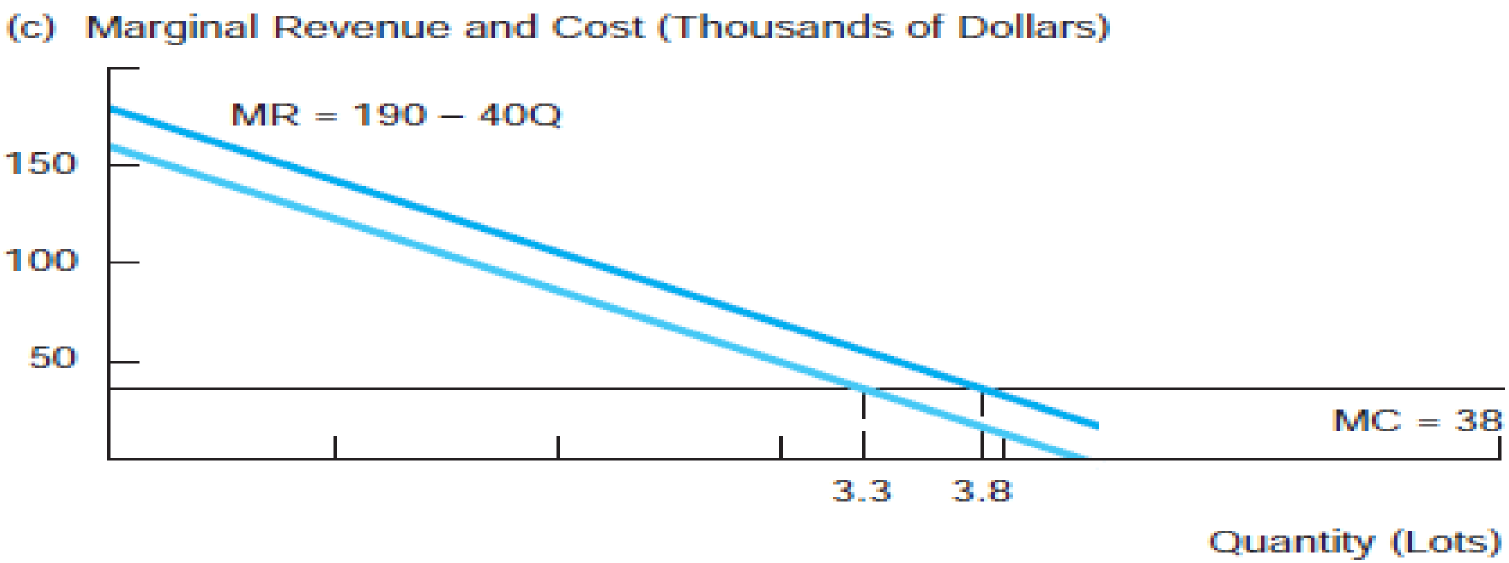


- What happens if economic conditions change? For instance if there is an increase in demand and increase in material cost, what will happen to the optimal price and output?



(a) Total Revenue, Cost, and Profit (Thousands of Dollars)
400
300
200
100
0
−100
Total cost
Revenue
Profit
0
2
4
6
8
(b) Marginal Revenue and Cost (Thousands of Dollars)
200
100
0
−100
−200
Marginal revenue
Marginal cost
Marginal profit
0
2
4
6
8
Quantity (Lots)


(b) Marginal Revenue and Cost (Thousands of Dollars)
150
100
50
MR = 170 – 40Q
MC = 46
MC = 38
3.1
3.3
Quantity (Lots)


(c) Marginal Revenue and Cost (Thousands of Dollars)
150
100
50
MR = 190 – 40Q
MC = 38
3.3
3.8
Quantity (Lots)

Exercises

1. A manager makes the statement that output should be expanded as long as average revenue exceeds average cost. Does this strategy make sense? Explain.

2. The original revenue function for the microchip producer is R 170Q - 20Q2. Derive the expression for marginal revenue, and use it to find the output level at which *revenue is maximized. Confirm that this is greater* than the firm’s profit-maximizing output, and explain why.

3. The college and graduate-school textbook market is one of the most profitable segments for book publishers. A best-selling accounting text—published by Old School Inc (OS)—has a demand curve: P = 150- Q, where Q denotes yearly sales (in thousands) of books. (In other words, Q = 20 means 20 thousand books.) The cost of producing, handling, and shipping each additional book is about $40, and the publisher pays a $10 per book royalty to the author. Finally, the publisher’s overall marketing and promotion spending (set annually) accounts for an average cost of about $10 per book.

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a). Determine OS's profit-maximizing output and price for the accounting text.

b). A rival publisher has raised the price of its best-selling accounting text by \$15. One option is to exactly match this price hike and so exactly preserve your level of sales. Do you endorse this price increase? (Explain briefly why or why not.)

c). To save significantly on fixed costs, Old School plans to contract out the actual printing of its textbooks to outside vendors. OS expects to pay a somewhat higher printing cost per book (than in part a) from the outside vendor (who marks up price above its cost to make a profit). How would outsourcing affect the output and pricing decisions in part (a)?